INDEFINITE INTEGRALS

PRIMITIVE OR ANTI DERIVATIVE

A function $\phi(x)$ is called a primitive or an anti derivative of a function f(x) if $\phi'(x) = f(x)$.

For example, $\frac{x^4}{4}$ is a primitive of x^3 because $\frac{d}{dx}\left(\frac{x^4}{4}\right) = x^3$.

Let ϕ (x) be a primitive of a function f(x) and let C be any constant. Then,

$$\frac{d}{dx}(\phi(x) + C) = \phi'(x) = f(x) \qquad [\because \phi'(x) = f(x)]$$

ϕ (x) + C is also a primitive of f(x).

Thus, if a function f(x) possesses a primitive, then it possesses infinitely many primitives which are contained in the expression $\phi(x)$ + C, where C is a constant.

For example, $\frac{x^4}{4}, \frac{x^4}{4} + 2, \frac{x^4}{4} - 1$ etc are primitives of x^3

INDEFINITE INTEGRAL

Let f(x) be a function. Then the collection of all its primitives is called the ***indefinite integral*** *of f(x) and is denoted by* $\int f(x)dx = \phi(x) + C$ *where $\phi(x)$ is primitive of f(x) and C is an arbitrary constant known as the* ***constant of integration.***

PROPERTIES OF INTEGRATION

1. $\int f_1(x) \pm f_2(x)\, dx = \int f_1(x)\,dx \pm \int f_2(x)\,dx$

2. $\int a\, f(x)\, dx = a \int f(x)\,dx$

3. $\int a\, dx = a\,x + C$

4. $\int 0 .dx =$ *constant*

5. $\frac{d\left(\int f(x)dx\right)}{dx} = f(x) + C$

	Derivative	Integration
(i)	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, \quad n \neq -1$	$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, n \neq 1$
(ii)	$\frac{d}{dx}(\log x) = \frac{1}{x}$	$\int (1/x)dx = \log \lvert x \rvert + C$
(iii)	$\frac{d}{dx}(e^x) = e^x$	$\int e^x\, dx = e^x + c$
(iv)	$\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x, a > 0, a \neq 1$	$\int a^x\, dx = \frac{a^x}{\log a} + C$
(v)	$\frac{d}{dx}(-\cos x) = \sin x$	$\int \sin x\, dx = -\cos x + C$
(vi)	$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\, dx = \sin x + c$
(vii)	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\, dx = \tan x + C$
(viii)	$\frac{d}{dx}(-\cot x) = \text{cosec}^2 x$	$\int \text{cosec}^2 x\, dx = -\cot x + C$
(ix)	$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x\, dx = \sec x + C$
(x)	$\frac{d}{dx}(-\text{cosec}\, x) = \text{cosec}\, x \cot x$	$\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + c$
(xi)	$\frac{d}{dx}(\log \sin x) = \cot x$	$\int \cot x\, dx = \log \lvert \sin x \rvert + C$
(xii)	$\frac{d}{dx}(-\log \cos x) = \tan x$	$\int \tan x\, dx = -\log \lvert \cos x \rvert + C$
(xiii)	$\frac{d}{dx}(\log(\sec x + \tan x)) = \sec x$	$\int \sec x\, dx = \log \lvert \sec x + \tan x \rvert + C$
(xiv)	$\frac{d}{dx}(\log(\text{cosec}\, x - \cot x)) = \text{cosec}\, x$	$\int \text{cosec}\, x\, dx = \log \lvert \text{cosec}\, x - \cot x \rvert + C$
(xv)	$\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}$	$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
(xvi)	$\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right) = \frac{1}{a^2 + x^2}$	$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
(xvii)	$\frac{d}{dx}\frac{1}{a}\cot^{-1}\frac{x}{a} = -\frac{1}{a^2 + x^2}$	$\int \frac{1}{a^2 + x^2} dx = -\frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$
(xviii)	$\frac{d}{dx}\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) = \frac{1}{x\sqrt{x^2 - a^2}}$	$\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$
(xix)	$\frac{d}{dx}\frac{1}{a}\text{cosec}^{-1}\left(\frac{x}{a}\right) = -\frac{1}{x\sqrt{x^2 - a^2}}$	$\int -\frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\text{cosec}^{-1}\left(\frac{x}{a}\right) + C$

EXAMPLES:

1. $\int x^4\, dx$

Sol. $\int x^4\, dx = \frac{x^{4+1}}{4+1} + C = \frac{x^5}{5} + C$

2. $\int \sqrt{x}\, dx$

Sol. $\int \sqrt{x}\, dx = \int x^{1/2}\, dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{2}{3} x^{3/2} + C$

3. $\int 1/\sqrt{x}\, dx$

Sol. $\int 1/\sqrt{x}\, dx = \int x^{-1/2}\, dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = 2x^{1/2} + C$

4. $\int 1/x^3\, dx$

Sol. $\int 1/x^3\, dx = \int x^{-3}\, dx = \frac{x^{-3+1}}{-3+1} + C = -\frac{1}{2x^2} + C$

5. $\int a^{3\log_a x}\, dx$

Sol. $\int a^{3\log_a x}\, dx = \int a^{\log_a x^3}\, dx = \int x^3\, dx = \frac{x^{3+1}}{3+1} + C = \frac{x^4}{4} + C \qquad [\because a^{\log_a x} = x]$

6. $\int \frac{1}{\sin^2 x \cos^2 x}\, dx$

Sol. $\int \frac{1}{\sin^2 x \cos^2 x}\, dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\, dx = \int \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\, dx$

$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx = \tan x - \cot x + C.$

7. $\int \sqrt{1-\cos 2x}\, dx$

Sol. $\int \sqrt{1-\cos 2x} = \int \sqrt{2\sin^2 x}\, dx = \sqrt{2} \int \sin x\, dx = -\sqrt{2} \cos x + C$

8. $\int \frac{\sec x}{\sec x + \tan x}\, dx$

Sol. $\int \frac{\sec x}{\sec x + \tan x}\, dx = \int \frac{\sec x(\sec x - \tan x)}{(\sec x + \tan x).(\sec x - \tan x)}\, dx = \int \frac{\sec^2 x - \sec x \tan x}{\sec^2 x - \tan^2 x}$

$= \int \sec^2 x - \sec x \tan x\, dx = \tan x - \sec x + C.$

9. $\int \frac{x^4}{x^2+1}\, dx$

Sol. $\int \frac{x^4}{x^2+1}\, dx = \int \frac{x^4 - 1 + 1}{x^2+1}\, dx = \int \frac{x^4-1}{x^2+1} + \frac{1}{x^2+1}\, dx$

$= \int (x^2 - 1)\, dx + \int \frac{1}{x^2+1}\, dx = \frac{x^3}{3} - x + \tan^{-1} x + C$

10. $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$

Sol. $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx = \int \frac{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x}$

[using $a^3 + b^3 = (a + b)^3 - 3ab\,(a + b)$]

$= \int \frac{1 - 3\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx = \int \left(\frac{1}{\sin^2 x \cos x} - 3 \right) dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} - 3dx$

$= \int (\sec^2 x + \operatorname{cosec}^2 x - 3)dx = \tan x - \cot x - 3x + C.$

11. $\int \frac{x + 2}{(x + 1)^2} dx$

Sol. $\int \frac{x + 2}{(x + 1)^2} dx = \int \frac{x + 1 + 1}{(x + 1)^2} dx = \int \frac{x + 1}{(x + 1)^2} + \int \frac{1}{(x + 1)^2} dx = \int \frac{1}{x + 1} dx + \int (x + 1)^{-2} dx$

$= \log |x + 1| + \frac{(x + 1)^{-1}}{(-1)} + C = \log |x + 1| - \frac{1}{x + 1} + C.$

12. $\int \frac{8x + 13}{\sqrt{4x + 7}} dx$

Sol. $\int \frac{8x + 13}{\sqrt{4x + 7}} dx = \int \frac{8x + 14 - 1}{\sqrt{4x + 7}} dx = \int \frac{2(4x + 7) - 1}{\sqrt{4x + 7}} dx = \int 2\sqrt{4x + 7}\, dx - \int \frac{1}{\sqrt{4x + 7}} dx$

$= 2 \left\{ \frac{(4x + 7)^{3/2}}{4x\frac{3}{2}} \right\} \left\{ \frac{(4x + 7)^{1/2}}{4x\frac{1}{2}} \right\} + C = 1/3\,(4x + 7)^{3/2} - ½\,(4x + 7)^{1/2} + C$

13. $\int \cos^2 x\, dx$

Sol. $\int \cos^2 x dx = \int \frac{1 + \cos 2x}{2} dx = \frac{1}{2} \int (1 + \cos 2x)\, dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2} \right) + C$

METHODS OF INTEGRATION

There are four basic methods of Integration:

(I) Integration by substitution **(II) Integration by parts**

(III) Integration by partial fractions (PF) **(IV) Integration by reduction formula**

(I) INTEGRATION BY SUBSTITUTION

The method of evaluating an integral by reducing it to standard form by a proper substitution is called integration by substitution.

If $\phi(x)$ is a continuously differentiable function, then to evaluate integrals of the form $\int f(\phi(x))\phi'(x)dx$,

We substitute $\phi(x) = t$ and $\phi'(x)\,dx = dt$.

These substitutions reduce the above integral to $\int f(t)dt$.

After evaluating this integral we substitute back the value of t.

(a) INTEGRALS OF THE FORM $\int f(ax+b)dx$

Theorem: If $I = \int f(x)dx = \phi(x)$, then $\int f(ax+b)dx = \frac{1}{a}\phi(ax+b)$

Proof: Let $I = \int f(ax+b)dx$. Putting *ax + b = t, we get* $a\,dx = dt$ or $dx = \frac{1}{a}dt$.

$$\therefore\ I = \int f(ax+b)dx = \frac{1}{a}\int f(t)dt = \frac{1}{a}\phi(t) = \frac{1}{a}.\phi(ax+b)$$

> **Note:** If in place of x we have ax + b, then the same formula is applicable but we must divide by coefficient of x or derivative of (ax + b) i.e. a. Thus $\int f(ax+b)dx = \frac{1}{a}\varphi(ax+b)$ where $\int f(x)dx = \varphi(x)$

For example

(i) $\int (ax+b)^n dx = \frac{1}{a}.\frac{(ax+b)^{n+1}}{n+1} + C,\ n \neq -1$

(ii) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$

(iii) $\int \operatorname{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + C$

(b) INTEGRALS OF THE FORM $\int \frac{f'(x)}{f(x)}dx$

Theorem: $\int \frac{f'(x)}{f(x)}dx = \log\{f(x)\}$

Proof: Let $I = \int \frac{f'(x)}{f(x)}dx$. Putting *f(x) = t, we get f ' (x)dx = dt.*

$$\therefore\ I = \int \frac{1}{t}dt = \log t = \log\{f(x)\}$$

SOME MORE FORMULAE	
$\int \mathbf{tan\ x\ dx} = \log\|\sec x\| + C$ $= -\log\|\cos x\| + C$	$\int \mathbf{cot\ x\ dx} = \log\|\sin x\| + C$ $= -\log\|\operatorname{cosec} x\| + C$
$\int \mathbf{sec\ x\ dx} = \log\|\sec x + \tan x\| + C$ $= \log\left\|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\| + C$	$\int \mathbf{cosec\ x\ dx} = \log\|\operatorname{cosec} x - \cot x\| + C$ $= \log\left\|\tan\frac{x}{2}\right\| + C$

Proof: Let $I = \int \tan x\ dx$. Then $I = \int \frac{\sin x}{\cos x} dx$.

Putting cos x = t, we get –sin x dx = dt or dx = – dt / sin x.

$\therefore I = \int \frac{\sin x}{\cos x} \times -\frac{dt}{\sin x} = -\int \frac{1}{t} dt = -\log|t| + C = -\log|\cos x| + C.$

Hence, $\int \tan x\ dx = -\log|\cos x| + C$ Or $\int \tan x\ dx = \log|\sec x| + C$

☞ **NOTE:**

(i) $\int [f(x)]^n f'(x)\ dx = \frac{(f(x))^{n+1}}{n+1} + C, \quad n \neq 1$

(ii) $\int a^{f(x)} f'(x) dx = \frac{a^{f(x)}}{\log a} + C$

(iii) $\int e^{f(x)} f'(x) dx = e^{f(x)} + C.$

EXAMPLES:

14. $\int \frac{\sec^2 x}{3+\tan x} \mathbf{dx}$

Sol. Put tan x = t, so that $\sec^2 x\ dx = dt$

$\therefore \int \frac{\sec^2 x}{3+\tan x} dx = \int \frac{\sec^2 x}{3+t} \cdot \frac{dt}{\sec^2 x} = \int \frac{1}{3+t} dt = \log|3+t| + C = \log|3+\tan x| + C.$

15. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

Sol. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int \frac{dt}{t} = \log|t| + C = \log\left|e^x + e^{-x}\right| + C$

16. $\int \frac{\sin x}{\sin(x-a)} dx$

Sol. Put x – a = t so that dx = dt

$\therefore \int \frac{\sin x}{\sin(x-a)} dx = \int \frac{\sin(a+t)}{\sin t} dt = \int \frac{\sin a \cos t + \cos a \sin t}{\sin t} dt$

$= \sin a \int \cot t\ dt + \cos a \int 1.\ dt = \sin a.\ \log|\sin t| + t \cos a + C$

$= \sin a\ .\ \log|\sin(x-a)| + (x - a)\cos a + C$

17. $\int \frac{\sin(x-a)}{\sin x} dx$

Sol. $\int \frac{\sin(x-a)}{\sin x} dx = \int \frac{\sin x \cos a - \cos x \sin a}{\sin x} dx = \int \cos a \, dx - \int \sin a \cot x \, dx$

$= \cos a \int 1.\, dx - \sin a \int \cot x \, dx = x \cos a - \sin a \log |\sin x| + C$

18. $\int \frac{1}{\cos(x-a)\cos(x-b)} dx$

Sol. $\int \frac{1}{\cos(x-a)\cos(x-b)} dx = \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x-a)\cos(x-b)} dx$

$= \frac{1}{\sin(a-b)} \int \frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)} dx$

$= \frac{1}{\sin(a-b)} \int [\tan(x-b) - \tan(x-a)] \, dx$

$= \frac{1}{\sin(a-b)} [-\log_e|\cos(x-b)| + \log_e|\cos(x-a)|] + C$

$= \frac{1}{\sin(a-b)} \log_e \left|\frac{\cos(x-a)}{\cos(a-b)}\right| + C$

19. $\int \frac{\sqrt{2+\log x}}{x} dx$

Sol. Let $I = \int \frac{\sqrt{2+\log x}}{x} dx$. Put $2 + \log x = t$ so that $\frac{1}{x} dx = dt$ or $dx = x\, dt$.

$\therefore I = \int \frac{\sqrt{t}}{x} x \, dt = \int \sqrt{t} \, dt = \frac{t^{3/2}}{3/2} + C = \frac{2}{3} t^{3/2} + C = \frac{2}{3} (2 + \log x)^{3/2} + C$

> ☞ **To evaluate integrals of the form $\int \sin mx \cos nx \, dx$, $\int \sin mx \sin nx \, dx$, $\int \cos mx \cos nx \, dx$ and $\int \cos mx \sin nx \, dx$, we use the following trigonometrical identities.**
>
> - ☞ **2 sin A cos B = sin (A + B) + sin (A – B)**
> - ☞ **2 cos A sin B = sin (A + B) – sin (A – B)**
> - ☞ **2 cos A cosB = cos (A + B) + cos (A – B)**
> - ☞ **2 sin A sin B = cos (A – B) – cos (A + B)**

For example:

20. $\int \sin 3x \sin 2x \, dx$

Sol. $\int \sin 3x \sin 2x \, 2x \, dx = \frac{1}{2} \int 2\sin 3x \sin 2x \, dx = \frac{1}{2} \int (\cos x - \cos 5x) \, dx = \frac{1}{2} (\sin x - \frac{\sin 5x}{5})$

21. $\int \cos 2x \cos 4x \, dx$.

Sol. $\int \cos 2x \cos 4x \, dx = \frac{1}{2} \int 2 \cos 4x \cos 2x \, dx = \frac{1}{2} \int (\cos 6x + \cos 2x) \, dx = \frac{1}{2}\left[\frac{\sin 6x}{6} + \frac{\sin 2x}{2}\right] + C.$

☞ **SELECTION FOR PROPER SUBSTITUTIONS:**

There are no hard and fast rules for making suitable substitutions. It is the experience which guides us best for the selection of a proper substitution. However, some useful suggestions are given below: -

Integral	Condition	Substitutions
a) $\int \sin^n x\,dx$,	n is positive odd integer	Put cos x = t.
b) $\int \cos^n x\,dx$,	n is positive odd integer	Put sin x = t.
c) $\int \sec^n x\,dx$,	n is positive even integer	Put tan x = t.
d) $\int \operatorname{cosec}^n x\,dx$,	n is positive even integer	Put cot x = t.
e) $\int \sin^m x \cos^n x\,dx$	1. If m is odd 2. If n is odd 3.If both m and n are even	Put cos x = t, Put sin x = t Use De'moivre's theorem.
f) The integrand is a rational function of e^x		Put e^x = t.

DE-MOIVRE'S THEOREM

We can solve $\int \sin^m x\, dx$ or $\int \cos^n x\, dx$ by the help of De-Moivre's theorem, when m and n are integers (even or odd).

$z = \cos x + i \sin x \quad \therefore\ \frac{1}{z} = (\cos x - i \sin x)$

$z + \frac{1}{z} = 2 \cos x,\ \ z - \frac{1}{z} = 2i \sin x$

Also $z^n + \frac{1}{z^n} = 2 \cos nx$,

$z^n - \frac{1}{z^n} = 2i \sin nx$.

For Example- To find the value of $\int \sin^6 x\, dx$

$2i \sin x = z - \frac{1}{z} \qquad \therefore\ (2i \sin x)^6 = \left(z - \frac{1}{z}\right)^6$

$= z^6 - 6z^4 + 15z^2 - 20 + 15\frac{1}{z^2} - 6\frac{1}{z^4} + \frac{1}{z^6}$

$= \left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$

$= 2\,[\cos 6x - 6 \cos 4x + 15 \cos 2x - 10]$

$\therefore\ \int \sin^6 x\, dx = \frac{1}{(2i)^6} \cdot 2\int (\cos 6x - 6\cos 4x + 15\cos 2x - 10)dx$

$= -\frac{1}{32}\left[\frac{\sin 6x}{6} \quad \frac{6}{4}\sin 4x + \frac{15}{2}\sin 2x \quad 10x\right]$

EXAMPLE:

22. $\int \sin^2 x \cos^5 x\, dx$

Sol. Let I = $\int \sin^2 x \cos^5 dx$.

Here power of cos x is odd, so we substitute sin x = t $\Rightarrow$ cos x dx = dt or dx = $\frac{dt}{\cos x}$

$\therefore$ I = $\int t^2 \cos^5 x \frac{dt}{\cos x} = \int t^2(1 - \sin^2 x)^2\, dt = \int t^2(t - t^2)^2\, dt$

$= \int (t^2 - 2t^4 + t^6)dt = \frac{t^3}{3} \quad 2\frac{t^5}{5} + \frac{t^7}{7} + C = \frac{\sin^3 x}{3} \quad 2\frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + C$

SOME IMPORTANT SUBSTITUTIONS

Following are some substitutions useful in evaluating integrals.

EXPRESSION	SUBSTITUTION
$a^2 + x^2$	x = a tan θ or a cot θ
$a^2 - x^2$	x = a sin θ or a cot θ
$x^2 - a^2$	x = a sec θ or a cosec θ
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	x = a cos 2θ
$\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(x-\beta)}$	x = α cos² θ + β sin² θ

SOME IMPORTANT RESULTS (derived by the substitution given in the table)	
$\int \frac{1}{x^2 - a^2}\, dx$	$\frac{1}{2a} \log \frac{(x-a)}{(x+a)}$ *when x > a.*
$\int \frac{1}{a^2 - x^2}\, dx$	$\frac{1}{2a} \log \frac{a+x}{a-x}$ *when x < a*
$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx$	*log [x +* $\sqrt{x^2 + a^2}$ *] or* $\sinh^{-1} x/a$
$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$	*log [x +* $\sqrt{x^2 - a^2}$ *} or* $\cosh^{-1} x/a$

Proof: $\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \log \frac{(x-a)}{(x+a)}$ when x > a.

Put x = a sec θ ⇒ dx = a sec θ tan θ dθ

$\Rightarrow \int \frac{1}{x^2 - a^2}\, dx = \int \frac{a \sec\theta \tan\theta\, d\theta}{a^2 \tan^2\theta}$

$= \int 1/a\, \mathrm{cosec}\,\theta\, d\theta = -1/a \log(\mathrm{cosec}\,\theta + \cot\theta) = -1/a \log\left(x/\sqrt{x^2 \quad a^2} + a/\sqrt{x^2 \quad a^2}\right) + C$

$= -1/a \log \left(\frac{(x+a)^2}{(x \quad a)(x+a)}\right)^{1/2} + C = ½\, a \log \frac{(x \quad a)}{(x+a)} + C$

EXAMPLES:

23. $\int \frac{1}{9x^2 - 4} dx$

Sol. $\int \frac{1}{9x^2 - 4} dx = \frac{1}{9} \int \frac{1}{x^2 - (2/3)^2} dx = \frac{1}{9}.\frac{1}{2x\frac{2}{3}} \log \left| \frac{x \quad \frac{2}{3}}{x + \frac{2}{3}} \right| + C = \frac{1}{12} \log \left| \frac{3x \quad 2}{3x+2} \right| + C$

24. $\int \frac{1}{\sqrt{9-25x^2}}dx$

Sol. $\int \frac{1}{\sqrt{9-25x^2}}dx = \frac{1}{5}\int \frac{1}{\sqrt{\frac{9}{25}-x^2}}dx = \frac{1}{5}\int \frac{1}{\sqrt{\left(\frac{3}{5}\right)^2-x^2}}dx = \frac{1}{5}\sin^{-1}\left(\frac{x}{3/5}\right) + C = \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$

(II) <u>BASIC THEORY OF INTEGRATION BY PARTS</u>

<u>Theorem:</u> If u and v are two functions of x, then

$$\int 1\text{st } 2\text{nd } dx = \mathbf{1st} \int 2\text{nd } dx - \int \left(\frac{d}{dx}1\text{st} \int 2\text{nd } dx\right) dx$$

We choose the first function as the function which comes first in the word

I L A T E

Where

I - Inverse trigonometric function ($\sin^{-1} x$, $\cos^{-1}x$, $\tan^{-1} x$ etc)

L - Logarithmic functions

A - Algebraic functions

T - Trigonometric functions

E - Exponential functions

SOME MORE…..

1. $\int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

2. $\int \sqrt{a^2+x^2}\, dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\log|x+\sqrt{a^2-x^2}| + C.$

3. $\int \sqrt{x^2-a^2}\, dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + C.$

<u>Proof of 1.</u>

Let $I = \int \sqrt{a^2-x^2}dx.$ Integrating by parts, we get

$I = \int \sqrt{a^2-x^2}.1\,dx = \sqrt{a^2-x^2}.x - \int \frac{1}{2}(a^2-x^2)^{-1/2}(0-2x).x\,dx$

$= x\sqrt{a^2-x^2} - \int \frac{-x^2}{\sqrt{a^2-x^2}}dx = x\sqrt{a^2-x^2} - \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}dx$

$= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}dx + a^2\int \frac{1}{\sqrt{a^2-x^2}}dx$

$= x\sqrt{a^2-x^2} - I + a^2\sin^{-1}\left(\frac{x}{a}\right) + C$

$\therefore 2I = x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right) + C \Rightarrow I = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + C$

EXAMPLES:

25. $\int \sin^{-1} x \, dx$

Sol Put $\sin^{-1} x = t$ so that $x = \sin t$ and $dx = \cos t \, dt$

$\therefore \int \sin^{-1} x = \int t \cos t \, dt = t \sin t - \int 1.(\sin t)\, dt = t \sin t - \int \sin t \, dt = t \sin t + \cos t + C$

$= x \sin^{-1} x + \sqrt{1 \quad \sin^2 t} + C = x \sin^{-1} x + \sqrt{1 \quad x^2} + C$

26. $\int \sin\sqrt{x} \, dx$

Sol. Let $I = \int \sin\sqrt{x} \, dx$. Put $\sqrt{x} = t$ so that $\frac{1}{2\sqrt{x}} dx = dt$ or $dx = 2\sqrt{x}\, dt$

$\therefore \int \sin\sqrt{x}\, dx = \int \sin 2t \, dt = 2\int t \sin t \, dt = 2\left[t(-\cos t) - \int 1.(-\cos t)\, dt\right] = 2\left[-t\cos t + \int \cos t \, dt\right]$

$= 2[-t \cos t + \sin t] + C = -2 [\sqrt{x} \cos\sqrt{x} + \sin\sqrt{x}] + C.$

27. $\int x \log(1 + x) dx$

Sol. $\int x \log(x + 1)\, dx = \log(x + 1).\frac{x^2}{2} - \int \frac{1}{x+1}.\frac{x^2}{2} dx = \frac{x^2}{2} \log(x + 1) - \frac{1}{2}\int \frac{x^2}{x+1} dx$

$= \frac{x^2}{2} \log(x + 1) - \frac{1}{2}\int \frac{x^2 - 1 + 1}{x+1} dx$

$= \frac{x^2}{2} \log(x + 1) - \frac{1}{2}\int \frac{x^2 - 1}{x+1} + \frac{1}{x+1} dx = \frac{x^2}{2}\log(x + 1) - \frac{1}{2}\left[\int\left((x-1) + \frac{1}{x+1}\right)dx\right]$

$= \frac{x^2}{2} \log(x + 1) - \frac{1}{2}\left[\frac{x^2}{2} - x + \log|x+1|\right] + C$

28. $\int x \cot^{-1} x \, dx$

Sol. $\int x \cot^{-1} x \, dx = (\cot^{-1} x).\left(\frac{x^2}{2}\right) - \int \frac{-1}{1+x^2}.\frac{x^2}{2} = \cot^{-1} x + \frac{1}{2}\int \frac{x^2}{x^2+1} dx$

$= \frac{x^2}{2} \cot^{-1} x + \frac{1}{2} \frac{x^2 + 1 - 1}{x^2 + 1} dx = \frac{x^2}{2} \cot^{-1} x + \frac{1}{2}\int\left(1 - \frac{1}{x^2+1}\right)dx$

$= \frac{x^2}{2} \cot^{-1} x + \frac{1}{2} [x - \tan^{-1} x] + C$

SOME STANDARD INTEGRAL FORMS

☞ $\int e^x (f(x) + f'(x))dx = e^x f(x) + C.$

☞ $\int g(x)dx = g(x)$, then $\int g(x)(f(x) + f'(x))dx = g(x)f(x) + C$

☞ $\int e^{kx} \{(f(x) + f'(x)\}\, dx = e^{kx} f(x) + C$

EXAMPLES:

29. $\int e^x(\tan x + \log \sec x)\,dx$

Sol. $\int e^x(\tan x + \log \sec x)\,dx = \int \underset{II}{e^x} \underset{I}{\log} \sec x dx\, dx + \int e^x \tan x\, dx$

$= (\log \sec x)e^x - \int \frac{1}{\sec x} . \sec x \tan x\, e^x\, dx + \int e^x \tan x\, dx + C$

$= e^x \log \sec x - \int e^x \tan x\, dx + \int e^x \tan x + C = e^x . \log \sec x + C$

30. $\int e^x\left(\frac{1+\sin x \cos x}{\cos^2 x}\right)dx$

Sol. $\int e^x\left(\frac{1+\sin x \cos x}{\cos^2 x}\right)dx = \int e^x\left(\frac{1}{\cos^2 x} + \frac{\sin x \cos x}{\cos^2 x}\right)dx = \int \; e^x(\sec^2 x + \tan x)dx = \; e^x \tan x + C$

31. $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx$

Sol. $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx = \frac{1}{x}e^x + C.$

32. $\int e^x\left(\frac{2+\sin 2x}{1+\cos 2x}\right)\,dx$

Sol. $\int e^x\left(\frac{2+\sin 2x}{1+\cos 2x}\right)\,dx = \int e^x\left(\frac{2+2\sin \cos x}{2\cos^2 x}\right)dx = \int e^x(\sec^2 x + \tan x)dx = \; e^x \tan x + C$

☛ **TWO IMPORTANT FORMULAE**

$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + C$

$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + C$

EXAMPLES:

33. $\int e^{2x} \sin 3x\, dx$

Sol. Let I = $\int e^{2x} \sin 3x$ dx.

Then I = $\int \underset{I}{e^{2x}} \underset{II}{\sin} 3x dx = e^{2x}\left(-\frac{\cos 3x}{3}\right) - \int 2e^{ex}\left(-\frac{\cos 3x}{3}\right)dx$

$\Rightarrow I = \frac{1}{3}e^{2x} \cos 3x + \frac{2}{3} \int \underset{I}{e^{2x}} \underset{II}{\cos} 3x dx$

$\Rightarrow I = -\frac{1}{3}e^{2x} \cos 3x + \frac{2}{3}\left[e^{2x}\frac{\sin 3x}{3} - \int 2e^{ex}\frac{\sin 3x}{3}dx\right]$

$\Rightarrow \; I = \frac{1}{3}e^{2x} \cos 3x + \frac{2}{9}e^{2x} \sin 3x - \frac{4}{9}\int e^{2x} \sin 3x\, dx$

$\Rightarrow \; I = -\frac{1}{3}\; e^{2x} \cos 3x + \frac{2}{9}e^{2x}\sin 3x - \frac{4}{9}I$

$\Rightarrow \quad I + \frac{4}{9}I = \frac{e^{2x}}{9}\,(2\sin 3x - 3\cos 3x)$

$\Rightarrow \quad \Rightarrow \frac{13}{9}I = \frac{e^{2x}}{9}\,(2\sin 3x - 3\cos 3x)$

$\Rightarrow I = \frac{e^{2x}}{13}\,(2\sin 3x - 3\cos 3x) + C$

(Or **use direct formula** i.e. $I = \frac{e^{2x}}{2^2 + 3^2}(2\sin 3x - 3\cos 3x)$)

34. $\int e^{-x} \cos x\, dx$

Sol. $\int e^{-x} \cos x\, dx = \frac{e^{-x}}{(-1)^2 + 1^2}(\sin x - \cos x) + C = \frac{e^{-x}}{2}(\sin x - \cos x) + C$

35. $\int \sin(\log x)\, dx$

Sol. Let $I = \int \sin(\log x)$. Put $\log x = t$ so that $1/x\, dx = dt \Rightarrow dx = x\, dt \Rightarrow dx = e^t\, dt$

$\therefore I = \int \underset{II}{\sin}\, \underset{I}{e^t}\, dt$

$I = \frac{e^t}{2}(\sin t - \cos t) + C.$

SPECIAL CASES

☞ **S-1 Integrals Of The Type**

A. $\int \frac{1}{ax^2 + bx + c} dx$

B. $\int \frac{1}{\sqrt{ax^2 + bx + c}}$

To evaluate this type of integrals we proceed as follows:

☞ **Step I:** Make the coefficient of x^2 unity by taking it common

☞ **Step II:** Add and subtract the square of half of the coefficient of x.

☞ **Step III:** Apply the formula discussed earlier.

EXAMPLES:

36. $\int \frac{1}{2x^2 + x - 1} dx$

Sol. $\int \frac{1}{2x^2 + x - 1} dx = \frac{1}{2} \int \frac{1}{x^2 + \frac{x}{2} - \frac{1}{2}} dx$

$= \frac{1}{2} \int \frac{1}{x^2 + x/2 + (1/4)^2 - (1/4)^2 - 1/2} dx = \frac{1}{2} \int \frac{1}{(x + 1/4)^2 - (3/4)^2} dx$

$= \frac{1}{2} \cdot \frac{1}{2(3/4)} \log \left| \frac{x + 1/4 - 3/4}{x + 1/4 + 3/4} \right| + C$

$= \frac{1}{3} \log \left| \frac{x - 1/2}{x + 1} \right| + C = \frac{1}{3} \log \left| \frac{2x + 1}{2(x + 1)} \right| + C$

37. $\int \frac{1}{x(x^5 + 1)} dx$

Sol. Let I = $\int \frac{1}{x(x^5 + 1)} dx$. Put $x^5 + 1 = t$, so that $5x^4 \, dx = dt$ or $dx = \frac{dt}{5x^4}$

$\therefore I = \int \frac{1}{xt} \frac{dt}{5x^4} = \frac{1}{5} \int \frac{1}{tx^5} dt = \frac{1}{5} \int \frac{1}{t^2 - t} dt$

$= \frac{1}{5} \int \frac{1}{t^2 - t + 1/4 - 1/4} dt = \frac{1}{5} \int \frac{1}{(t - 1/2)^2 - (1/2)^2}$

$= \frac{1}{5} \cdot \frac{1}{2(1/2)} \log \left| \frac{t - 1/2 - 1/2}{t - 1/2 + 1/2} \right| + C$

$= \frac{1}{5} \log \left| \frac{t - 1}{t} \right| + C = \frac{1}{5} \log \left| \frac{x^5}{x^5 + 1} \right| + C$

38. $\int \frac{1}{\sqrt{9+8x-x^2}}\,dx$

Sol. $\int \frac{1}{\sqrt{9+8x-x^2}}\,dx = \int \frac{1}{\sqrt{-\{x^2+8x-9\}}}\,dx = \int \frac{1}{\sqrt{-\{x^2-8x+16-25\}}}\,dx$

$= \int \frac{1}{\sqrt{-\{(x-4)^2-5^2\}}}\,dx = \int \frac{1}{\sqrt{5^2-(x-4)^4}}\,dx = \sin^{-1}\left(\frac{x-4}{5}\right) + C$

39. $\int \frac{1}{\sqrt{x(1-2x)}}\,dx$

Sol. $\int \frac{1}{\sqrt{x(1-2x)}}\,dx = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{x-2x^2}}\,dx$

$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-\left\{x^2-\frac{x}{2}+\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2\right\}}}\,dx = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-\left\{\left(x+\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2\right\}}}$

$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{1}{4}\right)^2-\left(x-\frac{1}{4}\right)^2}}\,dx = \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x-1/4}{1/4}\right) + C$

$= \frac{1}{\sqrt{2}} \sin^{-1}(4x-1) + C$

40. $\int \frac{1}{\sqrt{2x^2+3x-2}}\,dx$

Sol. $\int \frac{1}{\sqrt{2x^2+3x-2}}\,dx = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{x^2+\frac{3}{2}x-1}}\,dx$

$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(x-\frac{1}{3}\right)^2-\frac{9}{16}-1}}\,dx = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(x-\frac{1}{3}\right)^2-\left(\frac{5}{4}\right)^2}}\,dx$

$= \frac{1}{\sqrt{2}} \log\left|\left(x+\frac{3}{4}\right)+\sqrt{x^2+\frac{3}{2}x-1}\right| + C$

41. $\int \frac{e^x}{\sqrt{4-e^{2x}}}\,dx$

Sol. Let $I = \int \frac{e^x}{\sqrt{4-e^{2x}}}\,dx$. Put $e^x = t$ so that $e^x\,dx = dt$

$\therefore I = \int \frac{e^x}{\sqrt{4-t^2}} = \int \frac{e^x}{\sqrt{2^2-t^2}} = \sin^{-1}\left(\frac{t}{2}\right) + C = \sin^{-1}\left(\frac{e^x}{2}\right) + C$

42. $\int \frac{x^2}{\sqrt{1-x^6}}dx$

Sol. Let I = $\int \frac{x^2}{\sqrt{1-x^6}}dx$. Put $xx^3 = t$ so that $3x^2\,dx = dt$ or $dx = \frac{dt}{3x^2}$

$\therefore I = \frac{1}{3}\int \frac{x^2}{\sqrt{1-t^2}} = \frac{1}{3}\sin^{-1}(t) + C = \frac{1}{3}\sin^{-1}(x^3) + C$

43. $\int \frac{1}{x\sqrt{(\log x)^2 - 5}}dx$

Sol. Let I = $\int \frac{1}{x\sqrt{(\log x)^2 - 5}}$ dx Put log x = t, so that $\frac{1}{x}dx = dt$ or dx = xdt.

$\therefore I = \int \frac{dt}{\sqrt{t^2 - (\sqrt{5})^2}} = \log\left|t + \sqrt{t^2 - 5}\right| + C = \log\left|\log x + \sqrt{(\log x)^2 - 5}\right| + C$

44. $\sqrt{\frac{x}{a^3 - x^3}}dx$

Sol. $\sqrt{\frac{x}{a^3 - x^3}}dx = \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}}dx$.

Put $x^{3/2} = t$, so that $\frac{3}{2}x^{1/2}dx = dt$ or $\sqrt{x}\,dx = \frac{3}{2}dt$.

$\therefore I = \frac{2/3dt}{\sqrt{(a^{3/2})^2 - t^2}} = \frac{2}{3}\sin^{-1}\left(\frac{t}{a^{3/2}}\right) + C\ \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + C$

45. $\int \frac{1}{\sqrt{1-e^{2x}}}dx$

Sol. Let I = $\int \frac{1}{\sqrt{1-e^{2x}}}dx = \int \frac{1}{\sqrt{1-\frac{1}{e^{-2x}}}}dx = \int \frac{e^{-x}}{\sqrt{e^{-2x} - 1}}dx$

Put $e^{-x} = t$, so that $-e^{-x}dx = dt$

$\therefore I = -\int \frac{dt}{\sqrt{t^2 - 1^2}} = -\log\left|t + \sqrt{t^2 - 1}\right| + C = -\log\left|e^{-x} + \sqrt{e^{-2x} - 1}\right| + C$

☞ **S-2 Integrals of the Form** $\int \frac{px + q}{ax^2 + bx + c}dx$

To evaluate this type of integrals we express the numerator as follows:

px + q = λ (Diff. of denominator) + μ = λ (2ax + b) + μ

where λ and μ are constants to be determined by equating the coefficients of similar terms on both sides. So we have,

$\Rightarrow\ 2a\lambda = p$ and $b\lambda + \mu = \theta$

$\Rightarrow \lambda = \frac{p}{2a},\ \ \mu = q - \frac{bp}{2a}$

$\therefore \int \frac{px+q}{ax^2+bx+c}dx = \frac{p}{2a}\int \frac{2ax+b}{ax^2+bx+c}dx + \left(q-\frac{bp}{2a}\right)\int \frac{1}{ax^2+bx+c}dx$

$= \frac{p}{2a}\log\left|ax^2+bx+c\right| + \left(q-\frac{bp}{2a}\right)\int \frac{1}{ax^2+bx+c}dx$

The other integral on RHS can be evaluated by the method discussed in special case **S-1**

EXAMPLES:

46. $\int \frac{x}{x^2+x+1}dx$

Sol. Let $x = \lambda \frac{d}{dx}(x^2 + x + 1) + \mu$. Then $x = \lambda(2x + 1) + \mu$.

Comparing the coefficient of like powers of x, we get

$1 = 2\lambda$ and $\lambda + \mu = 0$

$\therefore \lambda = \frac{1}{2}$ and $\mu = -\lambda = -\frac{1}{2}$.

So, $\int \frac{x}{x^2+x+1}dx = \int \frac{1/2(2x+1)-1/2}{x^2+x+1}dx$

$= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx - \frac{1}{2}\int \frac{1}{x^2+x+1}dx$

$= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx - \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$

$= \frac{1}{2}\log\left|x^2+x+1\right| - \frac{1}{2}.\frac{1}{(\sqrt{3}/2)}\tan^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right) + C$

$= \frac{1}{2}\log\left|x^2+x+1\right| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

47. $\int \frac{x^3+x}{x^4-9}dx$

Sol. Let $I = \int \frac{x^3+x}{x^4-9}dx = \int \frac{x^3}{x^4-9}dx + \int \frac{x}{x^4-9}dx = I_1 + I_2 + C$ (say,).

` Where $I_1 = \int \frac{x^3}{x^4-9}dx$ and $I_2 = \int \frac{x}{x^4-9}dx$.

Put $x^4 - 9 = t$, so that $4x^3\,dx = dt$.

$\therefore I_1 = \int \frac{x^3}{t}.\frac{dt}{4x^3} = \frac{1}{4}\log|t| = \frac{1}{4}\log\left|x^4-9\right|$

$I_2 = \int \frac{x}{x^4-9}dx = \int \frac{x}{(x^2)^2-3^2}dx$. Put $x^2 = t$, so that $2x\,dx = dt$

$\therefore I_2 = \frac{1}{2}\int \frac{dt}{t^2-3^2} = \frac{1}{2}.\frac{1}{2x3}\log\left|\frac{t-3}{t+3}\right| = \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right| + C$

Hence, $I = \frac{1}{4}\log\left|x^4-9\right| + \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right| + C$

S-3 Integrals of the Form

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$$

To evaluate this type of integrals we express the numerator as follows:

px + q = λ(Diff .of denominator) + μ = λ(2ax + b) + μ

Where λ and μ are constants to be determined by equating the coefficients of similar terms on both sides. So we have

$2a\lambda = p$ and $b\lambda + \mu = q$

$\Rightarrow \lambda = \frac{p}{2a}$ and $\mu = q - \frac{bp}{2a}$

$$\therefore \int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx = \frac{p}{2a}\int \frac{pax+b}{\sqrt{ax^2+bx+c}}\,dx + \left(q-\frac{bp}{2a}\right)\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

$$= \frac{p}{a}\sqrt{ax^2+bx+c} + \left(q-\frac{bp}{2a}\right) \quad \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

EXAMPLES:

48. $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\,dx$

Sol. Let $2x + 3 = \lambda . \frac{d}{dx}(x^2 + 4x + 1) + \mu.$

Then $2x + 3 = \lambda\,(2x + 4) + \mu$

Comparing the coefficient of like power of x, we get

$2\lambda = 2$ and $4\lambda + \mu = 3 \quad \Rightarrow \lambda = 1$ and $\mu = -1$

$$\therefore \int \frac{2x+3}{\sqrt{x^2+4x+1}}\,dx = \int \frac{(2x+4)-1}{\sqrt{x^2+4x+1}}\,dx$$

$$= \int \frac{2x+4}{\sqrt{x^2+4x+1}}\,dx - \int \frac{1}{\sqrt{x^2+4x+1}}\,dx$$

$$= \int \frac{dt}{\sqrt{t}} - \int \frac{1}{\sqrt{(x+2)^2-(\sqrt{3})}}\,dx, \text{ where } t = x^2 + 4x + 1$$

$$= 2\sqrt{t} - \log\left|(x+2)+\sqrt{x^2+4x+1}\right| + C$$

$$= 2\sqrt{x^2+4x+1} - \log\left|x+2+\sqrt{x^2+4x+1}\right| + C$$

INTEGRALS OF THE T- RATIO

☞ **S-4 Integrals of the form**

$\int \frac{1}{a\sin^2 x + b\cos^2 x}\,dx,$

$\int \frac{1}{a + b\sin^2 x}\,dx,$

$\int \frac{1}{a + b\cos^2 x}\,dx,$

$\int \frac{1}{(a\sin x + b\cos x)^2}\,dx,$

$\int \frac{1}{a + b\sin^2 x + c\cos^2 x}\,dx$

To evaluate this type of integrals we proceed as follows:

Step I: Divide numerator and denominator both by $\cos^2 x$.
Step II: Replace $\sec^2 x$, if any, in denominator by $1 + \tan^2 x$
Step III: Put $\tan x = t$ so that $\sec^2 x\,dx = dt$.

After employing these steps the integral will reduce to the form $\int \frac{1}{at^2 + bt + c}\,dt$ which can be evaluated with the method discussed earlier.

EXAMPLES:

49. $\int \frac{\sin x}{\sin 3x}\,dx$

Sol. Let $I = \int \frac{\sin x}{\sin 3x}\,dx = \int \frac{\sin x}{3\sin x - 4\sin^3 x}\,dx = \int \frac{1}{3 - 4\sin^2 x}\,dx$

[Dividing numerator and denominator by $\cos^2 x$. Put $\tan x = t$ so that $\sec^2 x\,dx = dt$}

$\therefore I = \int \frac{dt}{3(1+t^2) - 4t^2} = \int \frac{dt}{3 - t^2} = \int \frac{1}{(\sqrt{3})^2 - t^2}\,dt$

$= \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| + C = \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + C$

☞ **S-5 Integrals of the form**

$\int \frac{1}{a\sin x + b\cos x}\,dx$

$\int \frac{1}{a + b\sin x}\,dx$

$\int \frac{1}{a + b\cos x}\,dx$

$\int \frac{1}{a\sin x + b\cos x + c}\,dx$

To evaluate this type of integrals we proceed as follows:

Step I: Put $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$, $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$

Step II: Replace $1 + \tan^2 \frac{x}{2}$ in the numerator by $\sec^2 \frac{x}{2}$

Step III: Put $\tan \frac{x}{2} = t$, so that $\frac{1}{2}\sec^2 \frac{x}{2}\,dx = dt$

After performing these three steps the integral reduces to the form $\int \frac{1}{at^2 + bt + c}\,dt$ which can be evaluated by methods discussed earlier.

EXAMPLES:

50. $\int \frac{1}{2+\cos x}\,dx$

Sol. Let I = $\int \frac{1}{2+\cos x}\,dx$. Putting cos x = $\frac{1-\tan^2 x/2}{1+\tan^2 x/2}$, we get

$$I = \int \frac{1}{2+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}}\,dx = \int \frac{1+\tan^2 x/2}{2(1+\tan^2 x/2)+1-\tan^2 x/2}\,dx = \int \frac{\sec^2 x/2}{\tan^2 x/2+3}\,dx$$

Put tan x/2 = t , so that (1/2) $\sec^2$ (x/2) = dt or $\sec^2$ (x/2) dx = 2dt.

$$\therefore I = \int \frac{2dt}{t^2+3} = 2\int \frac{dt}{t^2(\sqrt{3})^2}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x/2}{\sqrt{3}}\right) + C$$

☞ **Alternative method to evaluate integrals of the form** $\int \frac{1}{a\sin x+b\cos x}\,dx$

To evaluate this type of integrals we substitute

a = r cos θ, b = r sin θ

and so, r = $\sqrt{a^2+b^2}$, θ = $\tan^{-1}\left(\frac{b}{a}\right)$.

$\therefore$ a sin x + b cos x = r cos θ sin x + r sin θ cos x = r sin (x + θ)

$$\text{So, } \int \frac{1}{a\sin x+b\cos x}\,dx = \frac{1}{r}\int \frac{1}{\sin(x+\theta)}\,dx = \frac{1}{r}\int \operatorname{cosec}(x+\theta)\,dx$$

$$= \frac{1}{r}\log\left|\tan\left(\frac{x}{2}+\frac{\theta}{2}\right)\right| + C = \frac{1}{\sqrt{a^2+b^2}}\log\left|\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{b}{a}\right)\right| + C$$

51. $\int \frac{1}{\sin x+\sqrt{3}\cos x}\,dx$

Sol. Let 1 = r cos θ and √3 = r sin θ.

Then r = $\sqrt{1^2+(\sqrt{3})^2}$ = 2 and tan θ = $\frac{\sqrt{3}}{1}$ ⇒ θ = π/3

$$\therefore \int \frac{1}{\sin x+\sqrt{3}\cos x}\,dx = \frac{1}{r}\int \frac{1}{\sin x\cos\theta+\cos x\sin\theta}\,dx$$

$$= \frac{1}{r}\int \frac{1}{\sin(x+\theta)}\,dx = \frac{1}{r}\int \operatorname{cosec}(x+\theta)\,dx$$

$$= \frac{1}{r}\log\left|\tan\left(\frac{x}{2}+\frac{\theta}{2}\right)\right| + C = \frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{2}\right)\right| + C$$

☞ **S-6: Integrals of the form**

$$\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx$$

To evaluate this type of integrals we express the numerator as follows:

Numerator = λ(Diff. of denominator) + μ (denominator)

i.e, $(a \sin x + b \cos x) = \lambda \frac{d}{dx}(c \sin x + d \cos x) + \mu (c \sin x + d \cos x)$

where λ and μ are constants to be determined by comparing the coefficients of sin x and cos x on both sides

$$\therefore \int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx = \int \frac{\lambda(c\cos x - d\sin x) + \mu(c\sin x + d\cos x)}{c\sin x + d\cos x}$$

$$= \int \mu dx + \lambda \int \frac{a\sin x - b\cos x}{c\sin x + d\cos x}\,dx = \mu x + \lambda \log |c\sin x + d\cos x| + K$$

EXAMPLE:

52. $\int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx$

Sol. Let $I = \int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx.$

Let $3 \sin x + 2 \cos x = \lambda \frac{d}{dx}(3 \cos x + 2 \sin x) + \mu (3 \cos x + 2 \sin x)$

or $3 \sin x + 2 \cos x = \lambda (-\sin x + 2 \cos x) + \mu (3 \cos x + 2 \sin x)$

Comparing the coefficients of sin x and cos x on both sides, we get

$-3 \lambda + 2 \mu = 3$ and $2\lambda + 3\mu = 2$

$\Rightarrow \lambda = \frac{12}{3}$ and $\mu = -\frac{5}{13}$

$$\therefore I = \int \frac{12/13(-3\sin x + 2\cos x) - 5/13(3\cos x + 2\sin x)}{3\cos x + 2\sin x}\,dx$$

$$= -\frac{5}{13}\int 1.dx + \frac{12}{13}\int \frac{-3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx$$

$$= -\frac{5}{13}x + \frac{12}{3}\int \frac{dt}{t}, \text{ where } t = 3\cos x + 2\sin x$$

$$= -\frac{5}{13}x + \frac{12}{3}\log|t| + C = -\frac{5}{13}x + \frac{12}{3}\log|3\cos x + 2\sin x| + C$$

☞ **S-7 Integrals of the form**

$$\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx$$

To evaluate this type of integrals, we express the numerator as follows:

Numerator = λ (denominator) + μ (Diff. of denominator) + v

i.e., $(a \sin x + b \cos x + c) = \lambda (p \sin x + q \cos x + r) + \mu (p \cos x - q \sin x) + v$

where λ, μ, v are constants to be determined by comparing the coefficients of sin x, cos x and constant term on both sides.

$\therefore \int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}dx$

$= \int \lambda dx + \mu \int \frac{\text{Diff. of denominator}}{\text{denominator}}$ dx + v $\int \frac{1}{p\sin x + q\cos x + r}dx$

= $\lambda x + \mu \log |\text{denominator}| + v \int \frac{1}{p\sin x + q\cos x + r}dx$

The integral on RHS can be evaluated by the method discussed earlier.

EXAMPLE:

53. $\int \frac{3\cos x + 2}{\sin x + 2\cos + 3}dx$

Sol. Let I = $\int \frac{3\cos x + 2}{\sin x + 2\cos x + 3}dx$

Let 3 cos x + 2 = λ (sin x + 2 cos x + 3) + μ (cos x – 2 sin x) + v.

Comparing the coefficients of sin x, cos x and constant term on both sides,

We get $\lambda - 2\mu = 0$, $2\lambda + \mu = 3$, $3\lambda + v = 2$

$\Rightarrow \lambda = \frac{6}{5}, \mu = \frac{3}{5}$, and $v = -\frac{8}{5}$

$\therefore$ I = $\int \frac{\lambda(\sin x + 2\cos x + 3) + \mu(\cos x - 2\sin x) + v}{\sin x + 2\cos x + 3}dx$

$= \lambda \int dx + \mu \int \frac{\cos x - 2\sin x}{\sin x + 2\cos x + 3}dx$

$= \lambda x + \mu \log |\sin x + 2\cos x + 3| + v I_1$, where $I_1 = \int \frac{1}{\sin x + 2\cos x + 3}dx$

Putting, $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$, $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$, we get

$I_2 = \int \frac{1}{\frac{2\tan x/2}{1+\tan^2 x/2} + \frac{2(1-\tan^2 x2)}{1+\tan^2 x/2} + 3}dx$

$= \int \frac{1+\tan^2 x/2}{2\tan x/2 + 2 - 2\tan^2 x/2 + 3(1+\tan^2 x/2)}dx$

$= \int \frac{\sec^2 x/2}{\tan^2 x/2 + 2\tan x/2 + 5}dx$, Put $\tan\frac{x}{2} = t$ so that $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$ or $\sec^2\frac{x}{2}dx = 2dt$

$\therefore I_1 \int \frac{2dt}{t^2 + 2t + 5} = 2\int \frac{dt}{(t-1)^2 + 2^2} = \frac{2}{2}\tan^{-1}\left(\frac{t+1}{2}\right) = \tan^{-1}\left(\frac{\tan\frac{x}{2} + 1}{2}\right)$

Hence, I = $\lambda x + \mu \log |\sin x + 2\cos x + 3| + \tan^{-1}\left(\frac{\tan\frac{x}{2} + 1}{2}\right) + C$

where $\lambda = \frac{6}{5}$; $\mu = \frac{6}{5}$ and $v = \frac{8}{5}$

PARTIAL FRACTION

Integrals of form

$$\int \frac{P(x)}{ax^2 + bx + c}\,dx$$

Where P(x) is polynomial of degree greater than or equal to 2.

If $f(x)$ and g(x) are polynomials, then to evaluate $\int \frac{f(x)}{g(x)}dx$ adopt the following procedure.

(a) If deg $f(x) \geq$ deg g (x), then divide $f(x)$ by g(x). Let q(x) be the quotient and r(x), the remainder of this division, then $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$.

where q(x) being a polynomial can be integrated term by term and for integrating $\frac{r(x)}{g(x)}$, resolve this fraction into partial fractions.

The following table gives an idea of the types of partial fraction to be taken for different types of proper rational functions :

Types of proper rational functions	Types of partial fractions
$\frac{px+q}{(x-a)(x-b)}$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where x^2 + bx + c cannot be factorised	$\frac{A}{x-q} + \frac{Bx+C}{x^2+bx+c}$
$\frac{px^3+qx^2 rx+s}{(x^2+ax+b)(x^2+cx+d)}$ where x^2 + ax + b, x^3 + cx + d cannot be factorised.	$\frac{Ax+B}{x^2+ax+b} + \frac{Cx+D}{x^2+cx+d}$

54. Resolve $\frac{3x+2}{x^3-6x^2+11x-6}$ into partial fractions.

Sol. We have

$$\frac{3x+2}{x^3-6x^2+11x-6} = \frac{3x+2}{(x-1)(x-2)(x-3)}$$

Let $\frac{3x+2}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$. Then $\frac{3x+2}{(x-1)(x-2)(x-3)}$

$= \frac{A(x-2)(x-3) + B(x-1)(x-3) + c(x-1)(x-2)}{(x-1)(x-2) * x-3)}$

or 3x + 2 = A(x – 2) (x – 3) + B(x – 1) (x – 3) + C(x – 1) (x – 2) … (i)

Putting x = 1 = 0 or x = 1 in (i), we get

$$5 = A\,(1-2)\,(1-3) \Rightarrow A = \frac{5}{2}.$$

Putting x – 2 = 0 or x = 2 in (i), we obtain

$$8 = B\,(-1)\,(2-3) \Rightarrow B = -8.$$

Putting x – 3 = 0 or x = 3 in (i), we obtain

$$11 = C(3-1)\,(3-2) \Rightarrow C = \frac{11}{2}.$$

$$\therefore \quad \frac{3x+2}{x^3-6x^2+11x-6} = \frac{3x+2}{(x-1)(x-2)(x-3)}$$

$$= \frac{5}{2(x-1)} - \frac{8}{x-2} + \frac{11}{2(x-3)}$$

55. Resolve $\frac{2x}{x^3-1}$ into partial fractions.

Sol. Let $\frac{2x}{x^3-1} = \frac{2x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

Then $2x = A\,(x^2 + x + 1) + (Bx + C)\,(x-1)$ … (i)

Putting x – 1 = 0 or x = 1in (i), we get $2 = 3\,A \Rightarrow A = \frac{2}{3}$.

Putting x = 0 in (i), we get $A - C = 0 \Rightarrow C = A = \frac{2}{3}$.

Putting x = -1 in (i), we get $-2 = A + 2B - 2C$.

$$\Rightarrow \quad -2 = \frac{2}{3} - 2\,B - \frac{4}{3} \Rightarrow = \frac{2}{3}$$

$$\therefore \quad \frac{2x}{x^3-1} = \frac{2}{3}\cdot\frac{1}{x-1} + \frac{2/3x+2/3}{x^2+x+1}$$

or $$\frac{2x}{x^3-1} = \frac{2}{3}\cdot\frac{1}{x-1} + \frac{2}{3}\cdot\frac{x+1}{x^2+x+1}$$